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The Physical Neutrino Current from a Duality Transformation

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With a magnetic-like interpretation of the axial current, the polarization of (massless) neutrinos can be viewed as the result of a constrained duality transformation, set up so as to lead to an electromagnetic-like gauge field interaction.

1. INTRODUCTION

The apparent masslessness of neutrinos does not necessarily require them to be polarized (Sakurai, 1973). This paper attempts a simple justification of the polarization property, based on a magnetic-like interpretation of the conserved axial current. The massive Dirac equation is first reviewed in Section 2. Then, in Section 3 the massless case is considered. The polarization condition is obtained by means of a constrained duality transformation. The aforementioned constraint is set up so as to guarantee the existence of an electromagnetic-like gauge field: this is consistent with treating the neutrino situation as a limiting case of the massive one, and corresponds to the interaction with a (massless) Z^0 particle (Aitchison, 1982; Aitchison and Hey, 1989). The ambiguity of whether the neutrino polarization should be left-handed or right-handed remains.

All the treatment outlined in this paper is done before second quantization and before any possible symmetry-breaking effects (Aitchison, 1982; Aitchison and Hey, 1989). In particular, anomalies (Aitchison, 1982; Aitchison and Hey, 1989) are not a concern, as they only arise after second

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quantization. Notation is rather standard, with Greek (Latin) indices running through the values 0, 1, 2, 3 (1, 2, 3). The summation convention is applied to repeated up and down labels, and units are such that $\hbar = c = 1$.

2. DIRAC EQUATION

In a frame of reference of real spacetime coordinates $x = \{x^\mu\}$ and pseudo-Euclidean metric $g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$, the Dirac equation may be written as follows[2]:

$$i\gamma^\beta \partial_\beta \Psi(x) = m\Psi(x), \qquad m > 0 \tag{1}$$

Here, Ψ is a complex four-spinor and the Dirac matrices γ^μ obey the usual rules

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I, \qquad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{2}$$

with I being the 4×4 identity matrix. For convenience, we further define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$; this is Hermitian and unitary, and anticommutes with all the γ^μ. Also, we remark the alternative spacetime notation: $x = \{t, s\}$, with $t = x^0$, $s = \{x^k\}$.

The Dirac equation leads to a current

$$J^\mu(x) = q\bar{\Psi}(x)\gamma^\mu\Psi(x), \qquad \bar{\Psi}(x) = \Psi^\dagger(x)\gamma^0, \qquad q \in \mathbb{R} - \{0\} \tag{3}$$

which is conserved, that is, $\partial_\beta J^\beta = 0$. Besides, the zeroth component of J^μ/q is positive definite. On the other hand, the axial current

$$K^\mu(x) = \mathring{q}\bar{\Psi}(x)\gamma^\mu\gamma^5\Psi(x), \qquad \mathring{q} \in \mathbb{R} - \{0\} \tag{4}$$

is not conserved, except in the limit $m \to 0$. Both currents are real, and have rather obvious transformation properties under changes of coordinates $x \to x'$ (passive transformations). First of all, J^μ and K^μ behave like vectors under transformations of the proper Poincaré group. Then, for a spatial inversion ($t' = t, s' = -s$), J^μ is still a vector, while K^μ behaves like a pseudovector. Finally, in the case of a time inversion ($t' = -t, s' = s$), J^μ and K^μ behave, respectively, as a pseudovector and a vector. In fact, the

[2]For a review of the Dirac equation and most related topics, see, for instance, Sakurai (1973), Messiah (1966), Schweber (1961), Bogoliubov and Shirkov (1959), Bjorken and Drell (1964), and Itzykson and Zuber (1980). More specifically about gauge theory see Aitchison (1982) and Aitchison and Hey (1989).

appropriate time inversion transformation for spinors may be expressed as[3]

$$\Psi'(x') = (\text{global phase factor})\gamma^0\gamma^5\Psi(x) \tag{5}$$

A simplification is obtained by expediently requiring q and $\mathring{q}$ to be odd under time inversion[4] (rather than true scalars, as previously assumed): then, J^μ is a true vector and K^μ a true pseudovector under the full Poincaré group. This will be the convention adopted in the following, with the ratio $\mathring{q}/q$ remaining a true scalar. Henceforth, the symbols q_+ and $\mathring{q}_+$ shall indicate the values of q and $\mathring{q}$ in frames with t "running forward" [e.g., as established by comparison with the macroscopic arrow of time (Kittel, 1969)]. For the sake of clarity, we note that all the spinor transformations here considered are realized as follows:

$$\Psi'(x') = \Lambda\Psi(x), \qquad \Lambda^\dagger = \pm\gamma^0\Lambda^{-1}\gamma^0 \tag{6}$$

$$i\gamma^\beta\partial'_\beta\Psi'(x') = m\Psi'(x') \tag{7}$$

where the positive (negative) sign in equation (6) applies to the orthochronous (antiorthochronous) cases. Equations (6) and (7) refer to real linear changes of coordinates

$$x'^\mu = \lambda^\mu{}_\alpha x^\alpha + b^\mu, \qquad \lambda^\mu{}_\alpha g^{\alpha\beta}\lambda^\nu{}_\beta = g^{\mu\nu} \qquad \text{(Poincaré group)} \tag{8}$$

which all have $\det(\lambda^\mu{}_\nu) = \pm 1$. A pseudoscalar (pseudovector, pseudotensor) differs from a scalar (vector, tensor) by the presence of the aforementioned Jacobian determinant in the transformation equations. For instance, $\phi'(x') = [\det(\lambda^\mu{}_\nu)]\phi(x)$ is a pseudoscalar transformation.

As in §64 of Pauli (1981), the conservation of the vector current J^μ may be taken to imply the existence of a real antisymmetric tensor $F^{\mu\nu}$ such that

$$\partial_\beta F^{\beta\mu}(x) = J^\mu(x) \tag{9}$$

The tensor $F^{\mu\nu}$ is not necessarily a physical field. However, since the knowledge of J^μ gives no information about the divergence of the dual $*F^{\mu\nu}$, we can further request

$$\partial_\beta \, {*F^{\beta\mu}}(x) = 0 \tag{10}$$

[3]Time inversion as here introduced is just another passive transformation of coordinates (similar to spatial inversion), it is not time reversal! See, for instance, equation (6.35) of Bogoliubov and Shirkov (1959); beware of a misprint on the first line (a missing negative sign in front of x^0). Compare equation (XX.129) of Messiah (1966).

[4]More precisely, for any Poincaré transformation of coordinates, $q' = [\text{sign}(\partial t'/\partial t)]q$ and similarly for $\mathring{q}$. For electromagnetic-like quantities, this type of behavior is a legitimate convention, albeit not the most common one. See equations (9), (10) and equations (9), (15) in the following, and compare §6.11 and §6.12 of Jackson (1975).

claiming, then, that $F^{\mu\nu}$ of equations (9) and (10) is "physical". Due to equation (11) below, this is in agreement with the principles of gauge theories (see footnote 2). The dual $*F^{\mu\nu}$ is a pseudotensor, according to the definition $*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$, where ϵ is the totally antisymmetric (isotropic) symbol [equation 11.139 of Jackson (1975)]. As usual, equation (10) is readily solved by the introduction of a real vector potential A_μ (Jackson, 1975):

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \tag{11}$$

The corresponding gauge freedom relates to the following replacement (Hermitian operator → Hermitian operator)

$$\mathrm{i}\partial_\mu \to \mathrm{i}\mathscr{D}_\mu = \mathrm{i}\partial_\mu - q_+ A_\mu(x) \tag{12}$$

to be performed in equation (1). After the minimal replacement (12), the Dirac equation still leads to the conservation of the current (3), and q_+ plays the role of the coupling constant. The theory remains covariant [equations (6)–(8) with $\mathrm{i}\partial'_\mu \to \mathrm{i}\partial'_\mu - q_+ A'_\mu(x')$] and becomes gauge-invariant, as in §XX.20 of Messiah (1966).

In concluding this section, we find it convenient to define the "time index" T of a system of coordinates $\{t, s\}$ as $T = 0$ if t runs forward and $T = 1$ otherwise. Then

$$q = (-1)^T q_+, \qquad \mathring{q} = (-1)^T \mathring{q}_+ \tag{13}$$

Similarly, one may introduce the "space index" S as follows: $S = 0$ if the triplet $\{x^k\}$ is right-handed (e.g., as established by comparison with a human hand), and $S = 1$ otherwise.[5] For a generic four-spinor Θ, the statement

$$\gamma^5\Theta(x) = -(-1)^{T+S}\Theta(x) \tag{14}$$

defines it as a left-handed spinor, and does so in a coordinate-invariant manner: the factor $(-1)^{T+S}$ extends the validity of the definition to frames with $T + S = 1$, which are often ignored in the literature [compare equation (10.127) of Bjorken and Drell (1964), and related remarks]. A statement analogous to equation (14) can be made for right-handed spinors, multiplying by -1 the right-hand side of the same equation.

[5]We recall that one uses the right-hand (left-hand) rule in right-handed (left-handed) frames $\{x^k\}$. See p. 51 of Klein (1948) and §16.5 of Butkov (1968).

3. ZERO MASS

If one treats the $m = 0$ situation as a limiting case of equation (1), everything remains essentially the same as in the previous section, except that K^μ is now conserved: $\partial_\beta K^\beta = 0$ (see footnote 2). Once more, we can introduce a tensor $F^{\mu\nu}$ such that equation (9) is satisfied. Then, taking advantage of the remaining freedom, one is also able to request

$$\partial_\beta \; {}^*F^{\beta\mu}(x) = K^\mu(x) \tag{15}$$

This is, possibly, the most economical way of "utilizing" the two conserved currents. Again, $F^{\mu\nu}$ is a mathematical object, and may not be a physical field. However, seeking a gauge invariance like that of Section 2, we insist that a physical field $Z^{\mu\nu}$ be found by bringing (15) to the form of equation (10) by means of a duality transformation.[6] (That is, the gauge principle of Section 2 is here turned into a constraint acting on a set of duality-equivalent problems.) To that end, we write

$$\begin{aligned} Z^{\mu\nu}(x) &= F^{\mu\nu}(x) \cos\Delta + {}^*F^{\mu\nu}(x) \sin\Delta, \\ {}^*Z^{\mu\nu}(x) &= -F^{\mu\nu}(x) \sin\Delta + {}^*F^{\mu\nu}(x) \cos\Delta \end{aligned} \tag{16}$$

$$\begin{aligned} \mathscr{J}^\mu(x) &= J^\mu(x) \cos\Delta + K^\mu(x) \sin\Delta, \\ \mathscr{K}^\mu(x) &= -J^\mu(x) \sin\Delta + K^\mu(x) \cos\Delta = 0 \end{aligned} \tag{17}$$

with the parameter $\Delta \in [-\pi, \pi]$ being pseudoscalar.[7] This gives (with $Q = q/\cos\Delta$)

$$\partial_\beta Z^{\beta\mu}(x) = \mathscr{J}^\mu(x), \qquad \partial_\beta \; {}^*Z^{\beta\mu}(x) = 0 \tag{18}$$

$$\mathscr{J}^\mu(x) = Q\bar{\Psi}(x)\gamma^\mu\Psi(x) \tag{19}$$

and the equations $\mathscr{K}^\mu = 0$ read

$$\bar{\Psi}(x)\gamma^\mu[(-1)^{T+S}L_{++}I + \gamma^5]\Psi(x) = 0 \tag{20}$$

$$L = -(q \tan\Delta)/\mathring{q} = (-1)^{T+S}L_{++} \tag{21}$$

where L_{++} indicates the value of L in a system of coordinates with $S = 0$ and $T = 0$. The constraints (20) act as supplementary conditions to the massless Dirac equation (see footnote 2)

$$\gamma^\beta[i\partial_\beta - Q_+ Y_\beta(x)]\Psi(x) = 0, \qquad Q = (-1)^T Q_+ \tag{22}$$

[6]For a discussion of the duality transformation and its theoretical significance, see Jackson (1975).

[7]The massive case [that is, equations (9) and (10)] may be reproduced by means of $\mathring{q}$, $\Delta \to 0$ in equations (16)–(18).

which is here displayed after the appropriate minimal replacement (Q_+ plays the role of the coupling constant). The symbol Y_μ denotes the real vector potential of $Z_{\mu\nu}$, and equation (22) still leads to the conservation of the currents (3) and (4).

The formulation now consists of equations (18)–(20) and (22), with L_{++} and Q_+ as the new parameters (in place of q and $\mathring{q}$), and $Z^{\mu\nu}$ as the physical gauge field. In this context, the constraints (20) reduce to

$$\gamma^5\Psi(x) = -(-1)^{T+S}L_{++}\Psi(x) \tag{23}$$

under the standard assumption that the theory should be linear in the limit of a switched-off interaction ($Q_+ \to 0$). The above eigenvalue equation is clearly more restrictive than the original form (20). In particular, the only possible values of L_{++} in (23) are ± 1, with the upper (lower) sign defining a left-handed (right-handed) spinor. The general solution of (23) may be written as (see footnote 2)

$$\Psi(x) = \tfrac{1}{2}[I \mp (-1)^{T+S}\gamma^5]\Phi(x) \tag{24}$$

where Φ is an arbitrary four-spinor. Hence, equation (23) is eliminated by means of the replacement: $\Psi \to$ right-hand side of equation (24). The final formulation consists of (18), (24), and the following:

$$\mathscr{J}^\mu(x) = \frac{Q_+}{2}\bar{\Phi}(x)\,\gamma^\mu[(-1)^T I \mp (-1)^S\gamma^5]\Phi(x) \tag{25}$$

$$\gamma^\beta[i\partial_\beta - Q_+ Y_\beta(x)][(-1)^T I \mp (-1)^S\gamma^5]\Phi(x) = 0 \tag{26}$$

4. CONCLUSIONS

Aside from a sign ambiguity, the outlined procedure justifies the well-known polarization of neutrinos. In particular, equation (25) represents the correct physical neutrino current (upper sign).

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